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Non-canonical Poisson bracket for nonlinear elasticity with extensions to viscoelasticity

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Abstract. In this paper, we introduce a generalized bracket which is capable of generating the dynamical equations governing the flow of both elastic and viscoelastic media. This generalized bracket is divided into two parts: a non-canonical Poisson bracket and a new dissipation bracket. The non-canonical Poisson bracket is the Eulerian equivalent of the canonical Lagrangian Poisson bracket corresponding to an ideal (non-dissipative) continuum. It is derived for a nonlinear elastic medium, and then it is used to obtain the Eulerian equations of motion in nonlinear elasticity valid for large deformations. It is shown that the proposed non-canonical bracket naturally leads to a materially objective relation involving the upper-convected time derivative of a strain tensor, as suggested by Oldroyd 40 years ago. The dissipation bracket for linear irreversible thermodynamics is next proposed in a general, phenomenological circumstance, so that the dissipation processes occurring in real systems (viscous and relaxation phenomena) can be incorporated into the Hamiltonian formalism. This bracket diverges from previously proposed dissipation brackets in that it uses the same generating functional (i.e. the Hamiltonian) as the Poisson bracket, rather than an entropy functional or a dissipative potential. It is shown that, in combination with the choice of an appropriate Hamiltonian functional, the generalized bracket proposed here can generate the governing equations for many viscoelastic media, including the Voigt solid and the Maxwell viscoelastic fluid.

1. Introduction

The study of Hamiltonian mechanics has recently intensified due to increasing recognition of the wide applicability of these techniques. The Poisson bracket formulation of Hamiltonian dynamics, originally developed for optical and discrete particle systems during the previous century, found its major application in the development of quantum mechanics and quantum electrodynamics (Lanczos 1972). These ideas sparked little interest outside of the above-mentioned areas due to their apparent limitation, i.e. the description of non-dissipative, discrete systems. Renewed interest in the Poisson bracket formalism was generated when the ideas of Hamiltonian mechanics came to be applied to continuum systems, probably emanating from Arnold (1965, 1966, 1978). Only during the last three decades have researchers actively sought to extend these ideas to other, more complex, applications, such as continuum hydrodynamics, plasmas, etc. In particular, in the past decade much work has oriented towards the first non-canonical Poisson brackets developed for ideal fluids (Morrison and Greene 1980), Maxwell-Vlasov equations (Marsden and Weinstein 1982), etc, and their first applications in global stability analyses (Holm *et al* 1985).

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Moreover, there still exists a large body (indeed, the majority) of applications which cannot take advantage of this more general formulation of the dynamical equations in continuum mechanics using non-canonical Poisson brackets due to the presence of irreversible (dissipative) phenomena. Before the last few decades, an adequate theory of irreversible thermodynamics was not available and thus no one knew how to deal with the radical deviations from ideal system behaviour observed in dissipative systems. With the growth of the knowledge base concerning irreversible thermodynamics, it is thus only natural that scholars should return to Hamiltonian mechanics as a technique for both deriving useful dynamical equations as well as combining conservative and irreversible theories into a unified whole. Only a limited number of attempts to extend Hamiltonian methods to dissipative media have been reported in the literature (Kaufman 1984, Grmela 1984, 1985, 1989), none of which were entirely satisfactory.

In this paper, we wish to add to this growing body of knowledge by introducing what we call a generalized bracket (for lack of a better name), which can be used to express the more complex phenomena in Hamiltonian form; specifically, we wish to look at the cases of a nonlinear elastic medium and of a viscoelastic fluid. This generalized bracket is composed of two parts: a conservation part and an irreversible part. The conservative part is the well known Poisson bracket of traditional Hamiltonian mechanics described above; while the irreversible part is called a dissipation bracket. To be sure, the idea of a dissipation bracket is not entirely new, seeming to emanate from Kaufman (1984) and Grmela (1984). Thus we divide this paper into two parts, each dealing with one aspect of the generalized bracket.

In the study of hydrodynamics and elasticity, two different approaches have come down to us from the elders: the *Lagrangian or material description* and the *Eulerian or spatial description*. The material description concerns itself with specifying the path of a particular fluid particle of the continuum, of constant mass, in space and time; while the spatial approach describes the fluid properties at a specific location in space as functions of time. Although most of the recent work in Hamiltonian mechanics has been done in the material description, the complex systems are most aptly handled in the spatial description, for which we may use the machinery of irreversible thermodynamics to assist in the interpretation of dissipative phenomena.

In section 2, we start with the canonical, material Poisson bracket and derive from it explicitly the equivalent spatial Poisson bracket for nonlinear elasticity by a straightforward variable transformation. From this bracket, the ideal, spatial equations of motion are subsequently derived in their standard forms. In 1983, Holm and Kupershmidt used a Clebsch representation to arrive at an Eulerian Poisson bracket for elasticity (p 361); however, this bracket is not well suited for general rheological applications. The reasons for this are that their bracket is in terms of the deformation gradient tensor field, F, and that the resulting evolution equation for the structure does not meet the invariance criterion originally proposed by Oldroyd (1950). Oldroyd showed how the substantial time derivative changes when one moves from a convected coordinate to a fixed coordinate in order to maintain the proper frame-invariance relation.

Other researchers recently have attacked the problem of determining the non-canonical Poisson bracket for elasticity, most notably Marsden *et al* (1984) and Simo *et al* (1989). These derivations, being performed in the most generally applicable form (through differential geometry), are very difficult to follow. Therefore, we treat here a special case, that of the fixed-boundary viscoelastic fluid, where the derivation of the appropriate bracket may be accomplished through a formulation which better conforms to traditional techniques. This derivation is detailed in section 2 so that the reader may follow it without having to resort to any number of past articles to glean the necessary information. The reader is referred to the two citations above if interested in the more general derivation, which is in fact necessary if one wishes to study the more complex problems of rods, shells, etc.

After deriving the spatial forms of the Poisson brackets for the conservative part of the generalized bracket, we combine them in section 3 with the spatial dissipation bracket. In order to arrive at the proper, general dissipation bracket, we use as a guide the theory of linear (close to equilibrium) irreversible thermodynamics. The form for this dissipation bracket was recently introduced by Edwards and Beris (1990) and used to derive the simple hydrodynamic equations for both single-component and multicomponent fluid systems. The dissipation bracket which we propose differs from the type of dissipation bracket of Kaufman (1984, p 419) in that we use the same generating functional (i.e. the Hamiltonian) in both parts of the generalized bracket. This leads to a more aesthetically pleasing formulation since only one functional (the Hamiltonian) dictates the dynamics of both conservative and dissipative systems. Along the same lines, other researchers have previously reported work on dissipation brackets, most notably Grmela (1984, 1985, 1989), and we shall cite these as the occasion warrants. Finally, using the generalized bracket, we conclude by deriving the dissipative equations for a viscoelastic fluid in the spatial description, and showing two brief examples: the Voigt solid and the Maxwell viscoelastic fluid.

2. Non-canonical Poisson bracket for nonlinear elasticity

In order to derive the equations of motion for an ideal elastic medium in the spatial representation via the Poisson bracket, it is first necessary to arm ourselves with a few definitions. These are necessary when dealing with continuous systems (instead of the discrete systems for which the Poisson bracket was originally introduced) in order to construct the resulting equations, as well as to illustrate the analogy with the system of discrete particles.

Let us define an arbitrary functional, $F=F[a, b, \ldots]$, where $a, b, \ldots \in P$ (P being the operating space of the problem in consideration) are the dynamical variables of the system of interest, through the equation

$$F[a, b, \ldots] \equiv \int_{\Omega} f(a, b, \ldots)\, \mathrm{d}^3 x. \tag{2.1}$$

In this expression, f is a scalar function of the system variables with suitable continuity, Ω is the domain of interest with boundary $\partial\Omega$, represented by a spatial grid of points $\boldsymbol{x}$, and d^3x represents the appropriate volume element for the integration. In general, $a, b, \ldots$ depend on x and time, t. The above expression is not the most general form for the functional F, but is adequate for our purpose here.

With the above definition for F, we can also define the *Volterra functional derivatives* (for unconstrained systems) as

$$\frac{\delta F}{\delta a} \equiv \frac{\partial f}{\partial a} \qquad \frac{\delta F}{\delta b} \equiv \frac{\partial f}{\partial b} \qquad \ldots \in P. \tag{2.2}$$

Although the variables are treated as scalars here, for simplicity, they can be tensors of arbitrary order, in general, in which case the derivatives are with respect to the components of the tensor.

The situation is slightly more complex when one considers a functional of not only a, but ∇a, where $\nabla \equiv \partial/\partial \boldsymbol{x}$, i.e.

$$F[a, \nabla a] \equiv \int_\Omega f(a, \nabla a)\, \mathrm{d}^3x. \tag{2.3}$$

In this case, taking the variation of F, δF, gives

$$\delta F = \int_\Omega \left(\frac{\partial f}{\partial a} \delta a + \frac{\partial f}{\partial (\nabla a)} \cdot \delta(\nabla a) \right) \mathrm{d}^3x \tag{2.4}$$

whereupon switching the order of differentiation and integrating by parts the second term on the right-hand side yields

$$\delta F = \int_\Omega \left(\frac{\partial f}{\partial a} - \nabla \cdot \frac{\partial f}{\partial (\nabla a)} \right) \delta a\, \mathrm{d}^3x. \tag{2.5}$$

(Remember, the variations are defined to vanish on the boundary.) Therefore, the Volterra derivative of the functional F with respect to a becomes

$$\frac{\delta F}{\delta a} \equiv \frac{\partial f}{\partial a} - \nabla \cdot \frac{\partial f}{\partial (\nabla a)} \qquad \in P. \tag{2.6}$$

(This definition can easily be extended in a straightforward fashion to incorporate the dependence of F on higher-order derivatives of a.) Thus we see the intimate connection between the above functional derivatives and the calculus of variations, i.e. the functional derivative $\delta F/\delta a$ is just a notation indicating the Euler-Lagrange equation resulting from the variation of the functional F with respect to the variable a.

If a and b are functions of time, then we can write the total time derivative of F as

$$\begin{aligned} \frac{\mathrm{d}F}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t} \left(\int_\Omega f(a, \nabla a, b)\, \mathrm{d}^3x \right) \\ &= \int_\Omega \frac{\mathrm{d}}{\mathrm{d}t} f(a, \nabla a, b)\, \mathrm{d}^3x \\ &= \int_\Omega \left(\frac{\partial f}{\partial a} \frac{\partial a}{\partial t} + \frac{\partial f}{\partial (\nabla a)} \cdot \frac{\partial (\nabla a)}{\partial t} + \frac{\partial f}{\partial b} \frac{\partial b}{\partial t} \right) \mathrm{d}^3x \\ &= \int_\Omega \left(\frac{\delta F}{\delta a} \frac{\partial a}{\partial t} + \frac{\delta F}{\delta b} \frac{\partial b}{\partial t} \right) \mathrm{d}^3x. \end{aligned} \tag{2.7}$$

This expression is used throughout the paper.

Now that we have the above definitions, we are able to derive the non-canonical, spatial version of the Poisson bracket directly from the material, canonical Poisson bracket, and subsequently the spatial equations of motion. We follow a similar procedure to that proposed by Abarbanel *et al* (1988) based upon the more general formalism of Marsden *et al* (1984), although we have introduced some minor cosmetic changes: most notably that we are here treating the entropy density as the dynamical variable rather than the entropy per mass. This simplifies the final bracket expression and allows a straightforward extension to dissipative phenomena (see section 3).

As alluded to in the introduction, in the material description we can specify the equations of motion for a certain fluid particle of which we are following along its trajectory. This fluid particle of the continuous medium is labelled by a position (reference coordinate) vector at time $t=0$, $\boldsymbol{r}\in\mathcal{R}^3$, with a coordinate function $\boldsymbol{Y}(\boldsymbol{r},t)\in\mathcal{R}^3$ specifying the path of this particle in space and time. (For the sake of simplicity, we require both coordinates $\boldsymbol{r}$ and $\boldsymbol{Y}$ to be rectangular Cartesian.) At time $t=0$, the fluid occupies a region, Ω, with boundary $\partial\Omega$, and the initial condition on the fluid particle under consideration is obviously $\boldsymbol{Y}(\boldsymbol{r},0)=\boldsymbol{r}$. At the later time t, the fluid occupies the region Ω', with boundary $\partial\Omega'$, and the fluid particle is at position $\boldsymbol{Y}(\boldsymbol{r},t)$.

For simplicity in the following analysis, we consider the boundary of the fluid to be fixed, i.e. $\partial\Omega'=\partial\Omega$ for all times. In a subsequent work, we generalize this procedure for the free-boundary elastic medium, with applications to dissipative systems. In this section, we treat the system in terms of Ω and Ω' (it being understood that the two quantities are identical) so that most of the mathematics will carry over directly to the free-boundary problem, as represented by Abarbanel *et al* (1988) for ideal hydrodynamics of simple fluids. This simplification allows us to study the problem using a simple formalism similar to that of Abarbanel *et al* (1988) rather than the fully general methodology of Marsden *et al* (1984). In the case of a fixed boundary, all of the necessary boundary conditions are easily specified.

Associated with this fluid particle is a volume element at $t=0$, $\mathrm{d}^3r\equiv\mathrm{d}r_1\,\mathrm{d}r_2\,\mathrm{d}r_3$, where the vector $\boldsymbol{r}$ has components r_1, r_2 and r_3. Though the mass of the fluid particle must remain constant, its volume element is allowed to vary with time and space due to the compressible nature of the fluid. The volume element of the particle at any time t can be related to the volume element at $t=0$ through the *Euler relation* (Truesdell 1966, p 18),

$$\mathrm{d}^3Y=J\,\mathrm{d}^3r \tag{2.8a}$$

where J is the Jacobian which defines the mapping of the fluid particle from $\boldsymbol{r}$ at time $t=0$ to $\boldsymbol{Y}(\boldsymbol{r},t)$ at arbitrary time t. This Jacobian is given by the determinant of the *deformation gradient* tensor field, F, which, as the name suggests, measures the relative deformation undergone by the fluid element compared to the reference configuration:

$$J\equiv\det\mathsf{F}\qquad F_{\alpha\beta}\equiv\frac{\partial Y_\alpha}{\partial r_\beta}. \tag{2.8b}$$

Given the definitions (2.8b), it is a simple task to prove the *Boussinesq identity* (see, for example, Herivel 1955, p 347), which states that

$$\frac{\partial J}{\partial F_{\alpha\beta}}=J\frac{\partial r_\beta}{\partial Y_\alpha} \tag{2.9}$$

as well as the identities

$$\frac{\partial}{\partial r_\beta}\left(\frac{\partial J}{\partial F_{\alpha\beta}}\right)=0\qquad\text{and}\qquad\frac{\partial}{\partial Y_\alpha}\left(\frac{1}{J}\frac{\partial Y_\alpha}{\partial r_\beta}\right)=0. \tag{2.10}$$

The distribution of the system mass at $t=0$ can be described by a density function, $\rho_0=\rho_0(\boldsymbol{r})$. Since the mass of the fluid particle is conserved, the mass density at any $\boldsymbol{Y}(\boldsymbol{r},t)$, denoted $\rho(\boldsymbol{Y},t)$, must satisfy the conservation equation

$$\rho(\boldsymbol{Y},t)\,\mathrm{d}^3Y=\rho_0(\boldsymbol{r})\,\mathrm{d}^3r. \tag{2.11}$$

Thus the density can be written in terms of the Jacobian of equation (2.8*a*) as

$$\rho(\boldsymbol{Y}, t) = \frac{\rho_0(\boldsymbol{r})}{J}. \tag{2.12}$$

Hence we see that $\rho(\boldsymbol{Y}, t)$ depends on $\boldsymbol{Y}$ through the Jacobian or, more specifically, through the deformation of $\boldsymbol{Y}$ relative to the reference configuration, $\mathbf{F}$. Essentially, the functions $\rho_0(\boldsymbol{r})$ and $\rho(\boldsymbol{Y}, t) \in \mathscr{R}^+$, where $\mathscr{R}^+$ is the space of real, positive numbers.

In complex media, such as elastic solids, polymeric liquids and liquid crystals, one needs to describe not only the simple hydrodynamics of the fluid, but the internal structure as well. This structure evolves according to the kinematic history, and influences the internal stress of the material in addition to the overall hydrodynamic behaviour. Here we use a second-order deformation tensor to describe the internal structure of the elastic medium, which is a properly invariant tensor field, $\mathbf{c}(\boldsymbol{Y}, t)$. In general, the tensor field $\mathbf{c}(\boldsymbol{Y}, t)$ can be any arbitrary measure of the medium's structure; for instance, we could associate $\mathbf{c}$ with a weighted average over the usual orientational distribution function of kinetic theory (Bird *et al* 1987, p 63). Other interpretations also spring to mind. Here, we define $c_{\alpha\beta} \equiv F_{\alpha\gamma}F_{\beta\gamma}$ and wish to obtain the equations of motion of the ideal elastic medium in terms of this quantity.

The equations of motion in the material description for the fluid particle in consideration can be viewed as arising from a continuum Poisson bracket analogous to the Poisson bracket of discrete particle systems. In terms of the above functional derivatives, this bracket takes the form (Goldstein 1980, p 567)

$$\{F, G\}_{\mathrm{L}} \equiv \int_\Omega \left(\frac{\delta F}{\delta \boldsymbol{Y}} \cdot \frac{\delta G}{\delta \boldsymbol{\Pi}} - \frac{\delta F}{\delta \boldsymbol{\Pi}} \cdot \frac{\delta G}{\delta \boldsymbol{Y}} \right) \mathrm{d}^3 r \tag{2.13}$$

where the dynamical variables of the problem are the material coordinate, $\boldsymbol{Y}$, and the conjugate momentum of $\boldsymbol{Y}$, $\boldsymbol{\Pi}$, which is defined as $\boldsymbol{\Pi}(\boldsymbol{r}, t) = \rho_0(\boldsymbol{r}) \partial \boldsymbol{Y}(\boldsymbol{r}, t)/\delta t$ (Goldstein 1980, p 563). Note that the expression (2.13) is only correct for the simplified system with which we are dealing; i.e. a fixed-boundary system. F and G are arbitrary functionals of these two dynamical variables. The operating space, P, for this problem is then obviously

$$P \equiv \begin{cases} \boldsymbol{Y}(\boldsymbol{r}, t) \in \mathscr{R}^3 & \boldsymbol{Y}(\boldsymbol{r}, 0) = \boldsymbol{r} \text{ in } \Omega \\ \boldsymbol{\Pi}(\boldsymbol{r}, t) \in \mathscr{R}^3 & \boldsymbol{\Pi}(\boldsymbol{r}, 0) = \rho_0(\boldsymbol{r}) \partial \boldsymbol{Y}(\boldsymbol{r}, 0)/\partial t \qquad \text{in } \Omega. \end{cases} \tag{2.14}$$

The Hamiltonian (total energy) of this system, which arises through the Legendre transformation of the corresponding Lagrangian (Goldstein 1980, p 563), is written as

$$H[\boldsymbol{Y}, \boldsymbol{\Pi}] \equiv \int_\Omega \left(\frac{1}{2\rho_0} \boldsymbol{\Pi} \cdot \boldsymbol{\Pi} + \rho_0 e_{\mathrm{p}}^{\mathrm{v}}(\boldsymbol{Y}) + \rho_0 \hat{U}[\rho(\boldsymbol{Y}, t), \hat{S}(\boldsymbol{Y}, t), \mathbf{c}(\boldsymbol{Y}, t)] \right) \mathrm{d}^3 r \tag{2.15}$$

or, alternatively, as

$$H = \int_\Omega (e_{\mathrm{k}} + e_{\mathrm{p}} + u)\, \mathrm{d}^3 r \tag{2.16}$$

where $e_{\mathrm{p}}^{\mathrm{v}}$ is an external field potential and $\hat{U}$ is the internal energy per unit mass, the latter being a functional of the mass density, ρ, the specific entropy $\hat{S}$, and the structural parameter, $\mathbf{c}$. (Note that $\hat{S}$, the entropy per unit mass, is constant for the fluid particle, i.e. $\hat{S} = \hat{S}_0(\boldsymbol{r})$, adiabatic flow.) In equation (2.16), e_{k}, e_{p} and u are the kinetic, potential and internal energy densities (per unit reference volume), respectively.

The equations of motion now arise through the dynamical equation for arbitrary F,

$$\frac{\mathrm{d}F}{\mathrm{d}t}=\{F, H\}_{\mathrm{L}} \tag{2.17}$$

where we compare the Poisson bracket of (2.13) with the differentiation by parts of $\mathrm{d}F/\mathrm{d}t$:

$$\frac{\mathrm{d}F}{\mathrm{d}t}=\int_{\Omega}\left(\frac{\delta F}{\delta \boldsymbol{Y}}\cdot\frac{\partial \boldsymbol{Y}}{\partial t}+\frac{\delta F}{\delta \boldsymbol{\Pi}}\cdot\frac{\partial \boldsymbol{\Pi}}{\partial t}\right)\mathrm{d}^3 r. \tag{2.18}$$

Thus we have

$$\dot{Y}_\alpha(\boldsymbol{r}, t)=\frac{\delta H[\boldsymbol{Y}, \boldsymbol{\Pi}]}{\delta \Pi_\alpha(\boldsymbol{r}, t)}=\frac{\Pi_\alpha(\boldsymbol{r}, t)}{\rho_0(\boldsymbol{r})} \tag{2.19a}$$

and

$$\dot{\Pi}_\alpha(\boldsymbol{r}, t)=-\frac{\delta H[\boldsymbol{Y}, \boldsymbol{\Pi}]}{\delta Y_\alpha(\boldsymbol{r}, t)}=-\rho_0\frac{\partial e_{\mathrm{p}}^{v}}{\partial Y_\alpha}+\frac{\partial}{\partial r_\beta}\left(\rho_0\frac{\partial \hat{U}}{\partial F_{\alpha\beta}}\right). \tag{2.19b}$$

(Remember that $\hat{U}$ depends on ρ and $\mathbf{c}$, which depend in turn on the gradient of $\boldsymbol{Y}$, $\mathbf{F}$. Thus we need to invoke equation (2.6). Also note that $\hat{S}$, for a specific $\boldsymbol{r}$, is independent of $\boldsymbol{Y}$.)

Although equations (2.19) represent the equations of motion for the fluid particle, equation (2.19b) is not yet in a form which is immediately recognizable to students of hydrodynamics; however, using the standard thermodynamic definition of the pressure,

$$p\equiv\rho^2\left.\frac{\partial \hat{U}}{\partial \rho}\right|_{\hat{S}} \tag{2.20}$$

and the identities (2.9) and (2.10), equation (2.19b) can be rewritten as (see Seliger and Whitham (1968, p 4) for the simple fluid hydrodynamic case)

$$\rho\frac{\partial^2 Y_\alpha}{\partial t^2}=-\rho\frac{\partial e_{\mathrm{p}}^{v}}{\partial Y_\alpha}-\frac{\partial p}{\partial Y_\alpha}+\frac{\partial}{\partial Y_\gamma}\left(2\rho c_{\gamma\varepsilon}\frac{\partial \hat{U}}{\partial c_{\alpha\varepsilon}}\right). \tag{2.21}$$

Thus we have the usual equations of motion for an ideal fluid in the material description, with the third term on the right-hand side representing the elastic stress tensor.

Now that we have derived the dynamical equations in the material description, let us see how these quantities can be transformed into their spatial counterparts. In the spatial description, we consider a fixed-coordinate point in space (again assumed to be rectangular Cartesian convenience), $\boldsymbol{x}$, for which the (constant) volume element is given by d^3x. At this point, we sit and observe the properties of the fluid passing by us. For the transformation from material to spatial coordinates, we need to identify the fixed point $\boldsymbol{x}$ with the position function $\boldsymbol{Y}(\boldsymbol{r}, t)$, which coincide at the instant t. At this time, the fluid occupies the region Ω' with boundary $\partial\Omega'$. This procedure specifies a new function for all later times $\tilde{t}$, $\boldsymbol{R}(\boldsymbol{x}, \tilde{t})$, which maps $\boldsymbol{Y}$ back to the fixed position $\boldsymbol{x}$:

$$\boldsymbol{x}=\boldsymbol{Y}[\boldsymbol{R}(\boldsymbol{x}, \tilde{t}), \tilde{t}]. \tag{2.22}$$

Thus this function $\boldsymbol{R}(\boldsymbol{x}, \tilde{t})$ serves as a material label for the fluid particle which was at position $\boldsymbol{x}$ at time $\tilde{t}$, which is a determined function of the spatial field. Of course, $\boldsymbol{R}=\boldsymbol{r}$ at time t.

In the spatial description, we can now determine the dynamical variables for a nonlinear elastic medium as functions of x and t, in the domain Ω', with fixed boundary $\partial\Omega'$. Here the variables are the mass density, $\rho(x, t)$, the momentum density, $M(x, t) = \rho(x, t)v(x, t)$ (where $v(x, t)$ is the velocity vector field), the entropy density, $s(x, t) = \rho(x, t)\hat{S}(x, t)$, and the structure density, $\mathbf{C}(x, t) = \rho(x, t)\mathbf{c}(x, t)$. Thus the operating space, P, for the spatial description is

$$P \equiv \begin{cases} \rho(x, t) \in \mathscr{R}^+ & \rho(x, 0) = \rho_0(x) \text{ in } \Omega \\ M(x, t) \in \mathscr{R}^3 & M(x, 0) = M_0(x) \text{ in } \Omega, \text{ and } \nu \cdot M = 0 \text{ on } \partial\Omega' \\ \mathbf{C}(x, t) \in \mathscr{R}^3 \times \mathscr{R}^{3T} & \text{(appropriate boundary and initial conditions)} \\ s(x, t) \in \mathscr{R} & s(x, 0) = s_0(x) \text{ in } \Omega \end{cases} \tag{2.23}$$

where $\nu \cdot M = 0$ (ν being the outwardly directed unit vector normal to $\partial\Omega'$) ensures that the fluid does not penetrate the (fixed, in this case) boundary. Thus we see that the specification of a domain, Ω', with both initial and boundary conditions, is implicit to the spatial description, i.e. in a description based on x. The dynamical variables can now be written solely in terms of their material counterparts through the definition

$$\int_{\Omega'} g(b)\delta^3[b - a]\, \mathrm{d}^3 b \equiv \begin{cases} g(a) & b = a \\ 0 & b \neq a \end{cases} \tag{2.24}$$

where a and b are arbitrary vectors and g is an arbitrary function with suitable continuity. The mass density can be expressed as

$$\rho(x, \tilde{t}) = \rho(Y[R(x, \tilde{t}), \tilde{t}], \tilde{t})$$

or given that at time $\tilde{t} = t$:

$$\rho(x, t) = \rho(Y(r, t), t)$$

or via (2.24)

$$\rho(x, t) = \int_{\Omega'} \rho(Y(r, t), t)\delta^3[Y(r, t) - x]\, \mathrm{d}^3 Y$$

$$= \int_{\Omega} \rho_0(r)\delta^3[Y(r, t) - x]\, \mathrm{d}^3 r \tag{2.25}$$

where we have made use of equation (2.11). Similarly,

$$M(x, t) = \rho(Y(r, t), t)\dot{Y}(r, t)$$

$$= \int_{\Omega} \rho_0(r)\dot{Y}(r, t)\delta^3[Y(r, t) - x]\, \mathrm{d}^3 r$$

$$= \int_{\Omega} \Pi(r, t)\delta^3[Y(r, t) - x]\, \mathrm{d}^3 r \tag{2.26}$$

$$s(x, t) = \int_{\Omega} s_0(r)\delta^3[Y(r, t) - x]\, \mathrm{d}^3 r \tag{2.27}$$

and

$$C_{\alpha\beta}(x, t) = \int_{\Omega} \rho_0(r)c_{\alpha\beta}(Y(r, t), t)\delta^3[Y - x]\, \mathrm{d}^3 r \tag{2.28}$$

where we have multiplied both sides of equation (2.11) by $\hat{S}$:

$$s(\boldsymbol{Y}, t)\, \mathrm{d}^3 Y = s_0(\boldsymbol{r})\, \mathrm{d}^3 r. \tag{2.29}$$

Thus the spatial variables ρ, $\boldsymbol{M}$, s, and $\mathbf{C}$ are now expressed solely in terms of the material variables and it is apparent why we have chosen these as the natural variables (i.e. the density variables are necessary for the transformation from material to spatial coordinates). Note that the entropy density cannot be considered constant in either a material or a fixed-coordinate system.

Now we can follow the procedure of Marsden *et al* (1984), to derive the non-canonical, spatial equivalent of the Poisson bracket. Therefore, we need to define the functional derivatives of equation (2.13) through the chain rule of differentiation. For an arbitrary functional, $F[\rho(\boldsymbol{x}, t), \boldsymbol{M}(\boldsymbol{x}, t), s(\boldsymbol{x}, t), \mathbf{C}(\boldsymbol{x}, t)]$, following equation (2.7), these are

$$\frac{\delta F}{\delta Y_\alpha} = \int_{\Omega'} \left(\frac{\delta F}{\delta \rho(\boldsymbol{x}, t)} \frac{\delta \rho(\boldsymbol{x}, t)}{\delta Y_\alpha(\boldsymbol{r}, t)} + \frac{\delta F}{\delta M_\beta(\boldsymbol{x}, t)} \frac{\delta M_\beta(\boldsymbol{x}, t)}{\delta Y_\alpha(\boldsymbol{r}, t)} + \frac{\delta F}{\delta s(\boldsymbol{x}, t)} \frac{\delta s(\boldsymbol{x}, t)}{\delta Y_\alpha(\boldsymbol{r}, t)} + \frac{\delta F}{\delta C_{\gamma\beta}(\boldsymbol{x}, t)} \frac{\delta C_{\gamma\beta}(\boldsymbol{x}, t)}{\delta Y_\alpha(\boldsymbol{r}, t)} \right) \mathrm{d}^3 x \tag{2.30}$$

and

$$\frac{\delta F}{\delta \Pi_\alpha} = \int_{\Omega'} \left(\frac{\delta F}{\delta \rho(\boldsymbol{x}, t)} \frac{\delta \rho(\boldsymbol{x}, t)}{\delta \Pi_\alpha(\boldsymbol{r}, t)} + \frac{\delta F}{\delta M_\beta(\boldsymbol{x}, t)} \frac{\delta M_\beta(\boldsymbol{x}, t)}{\delta \Pi_\alpha(\boldsymbol{r}, t)} + \frac{\delta F}{\delta s(\boldsymbol{x}, t)} \frac{\delta s(\boldsymbol{x}, t)}{\delta \Pi_\alpha(\boldsymbol{r}, t)} + \frac{\delta F}{\delta C_{\gamma\beta}(\boldsymbol{x}, t)} \frac{\delta C_{\gamma\beta}(\boldsymbol{x}, t)}{\delta \Pi_\alpha(\boldsymbol{r}, t)} \right) \mathrm{d}^3 x. \tag{2.31}$$

In order for the integrals (2.30) and (2.31) to be proper inner products, the derivatives must all belong to the same operating space, in this case P (see definition (2.23)). Thus, $\boldsymbol{\nu} \cdot \delta F/\delta \boldsymbol{M} = 0$ on $\partial\Omega'$. Substitution of equations (2.30) and (2.31) into the Poisson bracket of equation (2.13) yields the spatial bracket for two arbitrary functionals $F[\rho(\boldsymbol{x}, t), \boldsymbol{M}(\boldsymbol{x}, t), s(\boldsymbol{x}, t), \mathbf{C}(\boldsymbol{x}, t)]$ and $G[\rho(z, t), \boldsymbol{M}(z, t), s(z, t), \mathbf{C}(\boldsymbol{x}, t)]$ (using, for convenience, an additional coordinate, z, which has the same properties as $\boldsymbol{x}$):

$$\begin{aligned}
\{F, G\}_{\mathrm{E}} = & \int_{\Omega'} \int_{\Omega'} \left(\frac{\delta F}{\delta \rho(\boldsymbol{x}, t)} \frac{\delta G}{\delta M_\beta(z, t)} - \frac{\delta G}{\delta \rho(\boldsymbol{x}, t)} \frac{\delta F}{\delta M_\beta(z, t)} \right) \\
& \times \{\rho(\boldsymbol{x}, t), M_\beta(z, t)\}_{\mathrm{L}}\, \mathrm{d}^3 z\, \mathrm{d}^3 x \\
& + \int_{\Omega'} \int_{\Omega'} \left(\frac{\delta F}{\delta s(\boldsymbol{x}, t)} \frac{\delta G}{\delta M_\beta(z, t)} - \frac{\delta G}{\delta s(\boldsymbol{x}, t)} \frac{\delta F}{\delta M_\beta(z, t)} \right) \\
& \times \{s(\boldsymbol{x}, t), M_\beta(z, t)\}_{\mathrm{L}}\, \mathrm{d}^3 z\, \mathrm{d}^3 x \\
& + \int_{\Omega'} \int_{\Omega'} \frac{\delta F}{\delta M_\gamma(\boldsymbol{x}, t)} \frac{\delta G}{\delta M_\beta(z, t)} \{M_\gamma(\boldsymbol{x}, t), M_\beta(z, t)\}_{\mathrm{L}}\, \mathrm{d}^3 z\, \mathrm{d}^3 x \\
& + \int_{\Omega'} \int_{\Omega'} \left(\frac{\delta F}{\delta C_{\alpha\beta}(\boldsymbol{x}, t)} \frac{\delta G}{\delta M_\gamma(z, t)} - \frac{\delta G}{\delta C_{\alpha\beta}(\boldsymbol{x}, t)} \frac{\delta F}{\delta M_\gamma(z, t)} \right) \\
& \times \{C_{\alpha\beta}(\boldsymbol{x}, t), M_\gamma(z, t)\}_{\mathrm{L}}\, \mathrm{d}^3 z\, \mathrm{d}^3 x \\
& + \int_{\Omega'} \int_{\Omega'} \frac{\delta F}{\delta C_{\alpha\beta}(\boldsymbol{x}, t)} \frac{\delta G}{\delta C_{\gamma\epsilon}(z, t)} \{C_{\alpha\beta}(\boldsymbol{x}, t), C_{\gamma\epsilon}(z, t)\}_{\mathrm{L}}\, \mathrm{d}^3 z\, \mathrm{d}^3 x
\end{aligned} \tag{2.32}$$

where $\{\cdot,\cdot\}_{\mathrm{L}}$ indicates the material bracket of equation (2.13). This expression was obtained through a lengthy (but straightforward) calculation, using the antisymmetry identities of the form

$$\{a, a\}_{\mathrm{L}} = 0 \tag{2.33a}$$

and

$$\{a, b\}_{\mathrm{L}} + \{b, a\}_{\mathrm{L}} = 0 \tag{2.33b}$$

where a and b represent the Eulerian dynamical variables. (Note that it is allowable to interchange the coordinates x and z under the integration.)

The five brackets $\{\cdot,\cdot\}_{\mathrm{L}}$ in equation (2.32) now need to be evaluated. From equations (2.25)–(2.28), we note that the necessary functional derivatives are

$$\frac{\delta\rho(\boldsymbol{x},t)}{\delta Y_\alpha(\boldsymbol{r},t)} = \rho_0(\boldsymbol{r})\frac{\partial\delta^3[\boldsymbol{Y}(\boldsymbol{r},t)-\boldsymbol{x}]}{\partial Y_\alpha(\boldsymbol{r},t)} \qquad \frac{\delta\rho(\boldsymbol{x},t)}{\delta\Pi_\alpha(\boldsymbol{r},t)} = 0 \tag{2.34}$$

$$\frac{\delta s(\boldsymbol{x},t)}{\delta Y_\alpha(\boldsymbol{r},t)} = s_0(\boldsymbol{r})\frac{\partial\delta^3[\boldsymbol{Y}(\boldsymbol{r},t)-\boldsymbol{x}]}{\partial Y_\alpha(\boldsymbol{r},t)} \qquad \frac{\delta s(\boldsymbol{x},t)}{\delta\Pi_\alpha(\boldsymbol{r},t)} = 0 \tag{2.35}$$

$$\begin{gathered}\frac{\delta M_\beta(\boldsymbol{x},t)}{\delta Y_\alpha(\boldsymbol{r},t)} = \Pi_\beta(\boldsymbol{r},t)\frac{\partial\delta^3[\boldsymbol{Y}(\boldsymbol{r},t)-\boldsymbol{x}]}{\partial Y_\alpha(\boldsymbol{r},t)}\\ \frac{\delta M_\beta(\boldsymbol{x},t)}{\delta\Pi_\alpha(\boldsymbol{r},t)} = \delta_{\alpha\beta}\delta^3[\boldsymbol{Y}(\boldsymbol{r},t)-\boldsymbol{x}]\end{gathered} \tag{2.36}$$

$$\begin{gathered}\frac{\delta C_{\alpha\beta}(\boldsymbol{x},t)}{\delta Y_\gamma(\boldsymbol{r},t)} = \rho_0 c_{\alpha\beta}\frac{\partial\delta^3[\boldsymbol{Y}-\boldsymbol{x}]}{\partial Y_\gamma(\boldsymbol{r},t)} - \frac{\partial}{\partial r_\eta}\left(\rho_0\delta^3[\boldsymbol{Y}-\boldsymbol{x}]\frac{\partial c_{\alpha\beta}}{\partial F_{\gamma\eta}}\right)\\ \frac{\delta C_{\alpha\beta}(\boldsymbol{x},t)}{\delta\Pi_\gamma(\boldsymbol{r},t)} = 0\end{gathered} \tag{2.37}$$

where $\boldsymbol{\delta}$ is the identity tensor. Consequently, we have $\{C_{\alpha\beta}(\boldsymbol{x},t), C_{\gamma\epsilon}(\boldsymbol{z},t)\}_{\mathrm{L}} = 0$,

$$\begin{aligned}\{\rho(\boldsymbol{x},t), M_\beta(\boldsymbol{z},t)\}_{\mathrm{L}} &= \int_\Omega\left(\frac{\delta\rho(\boldsymbol{x},t)}{\delta Y_\alpha(\boldsymbol{r},t)}\frac{\delta M_\beta(\boldsymbol{z},t)}{\delta\Pi_\alpha(\boldsymbol{r},t)} - \frac{\delta M_\beta(\boldsymbol{z},t)}{\delta Y_\alpha(\boldsymbol{r},t)}\frac{\delta\rho(\boldsymbol{x},t)}{\delta\Pi_\alpha(\boldsymbol{r},t)}\right)\mathrm{d}^3r\\ &= \int_\Omega \rho_0(\boldsymbol{r})\frac{\partial\delta^3[\boldsymbol{Y}(\boldsymbol{r},t)-\boldsymbol{x}]}{\partial Y_\alpha(\boldsymbol{r},t)}\delta_{\alpha\beta}\delta^3[\boldsymbol{Y}(\boldsymbol{r},t)-\boldsymbol{z}]\,\mathrm{d}^3r\\ &= \rho(\boldsymbol{z},t)\frac{\partial\delta^3[\boldsymbol{z}-\boldsymbol{x}]}{\partial z_\beta}\end{aligned} \tag{2.38}$$

$$\{s(\boldsymbol{x},t), M_\beta(\boldsymbol{z},t)\}_{\mathrm{L}} = s(\boldsymbol{z},t)\frac{\partial\delta^3[\boldsymbol{z}-\boldsymbol{x}]}{\partial z_\beta} \tag{2.39}$$

$$\{M_\gamma(\boldsymbol{x},t), M_\beta(\boldsymbol{z},t)\}_{\mathrm{L}} = M_\gamma(\boldsymbol{z},t)\frac{\partial\delta^3[\boldsymbol{z}-\boldsymbol{x}]}{\partial z_\beta} - M_\beta(\boldsymbol{x},t)\frac{\partial\delta^3[\boldsymbol{x}-\boldsymbol{z}]}{\partial x_\gamma} \tag{2.40}$$

and

$$\{C_{\alpha\beta}(\boldsymbol{x},t),M_\gamma(\boldsymbol{z},t)\}_{\mathrm{L}}$$

$$=\int_\Omega\left(\rho_0c_{\alpha\beta}\frac{\partial\delta^3[\boldsymbol{Y}-\boldsymbol{x}]}{\partial Y_\gamma}-\frac{\partial}{\partial r_\eta}(\rho_0\delta^3[\boldsymbol{Y}-\boldsymbol{x}])\frac{\partial c_{\alpha\beta}}{\partial F_{\gamma\eta}}\right)\delta^3[\boldsymbol{Y}-\boldsymbol{z}]\,\mathrm{d}^3\boldsymbol{r}$$

$$=\int_{\Omega'}\left(\rho c_{\alpha\beta}\frac{\partial\delta^3[\boldsymbol{Y}-\boldsymbol{x}]}{\partial Y_\gamma}-\frac{1}{J}\frac{\partial}{\partial r_\eta}(\rho J\delta^3[\boldsymbol{Y}-\boldsymbol{x}]\{F_{\alpha\eta}\delta_{\beta\gamma}+F_{\beta\eta}\delta_{\alpha\gamma}\})\right)$$

$$\times\delta^3[\boldsymbol{Y}-\boldsymbol{z}]\,\mathrm{d}^3Y$$

$$=\int_{\Omega'}\left(\rho c_{\alpha\beta}\frac{\partial\delta^3[\boldsymbol{Y}-\boldsymbol{x}]}{\partial Y_\gamma}-\frac{1}{J}\frac{\partial Y_\varepsilon}{\partial r_\eta}\frac{\partial}{\partial Y_\varepsilon}(\rho J\delta^3[\boldsymbol{Y}-\boldsymbol{x}]\{F_{\alpha\eta}\delta_{\beta\gamma}+F_{\beta\eta}\delta_{\alpha\gamma}\})\right)$$

$$\times\delta^3[\boldsymbol{Y}-\boldsymbol{z}]\,\mathrm{d}^3Y$$

$$=\int_{\Omega'}\left(\rho c_{\alpha\beta}\frac{\partial\delta^3[\boldsymbol{Y}-\boldsymbol{x}]}{\partial Y_\gamma}-\frac{\partial}{\partial Y_\varepsilon}(\rho F_{\varepsilon\eta}\delta^3[\boldsymbol{Y}-\boldsymbol{x}]\{F_{\alpha\eta}\delta_{\beta\gamma}+F_{\beta\eta}\delta_{\alpha\gamma}\})\right)$$

$$\times\delta^3[\boldsymbol{Y}-\boldsymbol{z}]\,\mathrm{d}^3Y$$

$$=C_{\alpha\beta}(\boldsymbol{z},t)\frac{\partial\delta^3[\boldsymbol{z}-\boldsymbol{x}]}{\partial z_\gamma}-\frac{\partial}{\partial z_\nu}(\delta^3[\boldsymbol{z}-\boldsymbol{x}]\{C_{\nu\alpha}(\boldsymbol{z},t)\delta_{\beta\gamma}+C_{\nu\beta}(\boldsymbol{z},t)\delta_{\alpha\gamma}\})\qquad(2.41)$$

where the second identity of (2.10) was used.

The brackets (2.38)–(2.41) can be substituted back into equation (2.32), which we rewrite now as

$$\{F,G\}_{\mathrm{E}}=\{F,G\}_{\mathrm{E}}^{\rho}+\{F,G\}_{\mathrm{E}}^{s}+\{F,G\}_{\mathrm{E}}^{M}+\{F,G\}_{\mathrm{E}}^{C}\qquad(2.42)$$

where the sub-brackets $\{F,G\}_{\mathrm{E}}^{a}$ have the obvious interpretation from equation (2.32). Let us look at each sub-bracket in turn. The first sub-bracket, $\{F,G\}_{\mathrm{E}}^{\rho}$, can be written as

$$\{F,G\}_{\mathrm{E}}^{\rho}=\int_{\Omega'}\int_{\Omega'}\left(\frac{\delta F}{\delta\rho(\boldsymbol{x},t)}\frac{\delta G}{\delta M_\beta(\boldsymbol{z},t)}-\frac{\delta G}{\delta\rho(\boldsymbol{x},t)}\frac{\delta F}{\delta M_\beta(\boldsymbol{z},t)}\right)$$

$$\times\rho(\boldsymbol{z},t)\frac{\partial\delta^3[\boldsymbol{z}-\boldsymbol{x}]}{\partial z_\beta}\,\mathrm{d}^3z\,\mathrm{d}^3x.\qquad(2.43)$$

Integrating this expression by parts with respect to $\boldsymbol{z}$ yields two terms:

$$\{F,G\}_{\mathrm{E}}^{\rho}=\int_{\Omega'}\int_{\partial\Omega'}\left(\frac{\delta F}{\delta\rho(\boldsymbol{x},t)}\frac{\delta G}{\delta M_\beta(\boldsymbol{z},t)}-\frac{\delta G}{\delta\rho(\boldsymbol{x},t)}\frac{\delta F}{\delta M_\beta(\boldsymbol{z},t)}\right)$$

$$\times\nu_\beta\delta^3[\boldsymbol{z}-\boldsymbol{x}]\,\mathrm{d}^2z\,\mathrm{d}^3x$$

$$-\int_{\Omega'}\int_{\Omega'}\left(\frac{\delta F}{\delta\rho(\boldsymbol{x},t)}\frac{\partial}{\partial z_\beta}\left\langle\rho(\boldsymbol{z},t)\frac{\delta G}{\delta M_\beta(\boldsymbol{z},t)}\right\rangle-\frac{\delta G}{\delta\rho(\boldsymbol{x},t)}\frac{\partial}{\partial z_\beta}\right.$$

$$\left.\times\left\langle\rho(\boldsymbol{z},t)\frac{\delta F}{\delta M_\beta(\boldsymbol{z},t)}\right\rangle\right)\delta^3[\boldsymbol{z}-\boldsymbol{x}]\,\mathrm{d}^3z\,\mathrm{d}^3x\qquad(2.44)$$

but we know that $\boldsymbol{\nu}\cdot\delta F/\delta\boldsymbol{M}=0$ on $\partial\Omega'$, so that the surface term vanishes (besides, the variations are required to vanish on the boundary). Therefore, only the bulk term survives, which after integrating over z becomes

$$\{F, G\}_{\mathrm{E}}^{\rho}=-\int_{\Omega'}\left(\frac{\delta F}{\delta\rho(\boldsymbol{x}, t)}\frac{\partial}{\partial x_\beta}\left\langle\rho(\boldsymbol{x}, t)\frac{\delta G}{\delta M_\beta(\boldsymbol{x}, t)}\right\rangle - \frac{\delta G}{\delta\rho(\boldsymbol{x}, t)}\frac{\partial}{\partial x_\beta}\left\langle\rho(\boldsymbol{x}, t)\frac{\delta F}{\delta M_\beta(\boldsymbol{x}, t)}\right\rangle\right)\mathrm{d}^3x. \tag{2.45}$$

In an analogous fashion, we can also write

$$\{F, G\}_{\mathrm{E}}^{s}=-\int_{\Omega'}\left(\frac{\delta F}{\delta s(\boldsymbol{x}, t)}\frac{\partial}{\partial x_\beta}\left\langle s(\boldsymbol{x}, t)\frac{\delta G}{\delta M_\beta(\boldsymbol{x}, t)}\right\rangle - \frac{\delta G}{\delta s(\boldsymbol{x}, t)}\frac{\partial}{\partial x_\beta}\left\langle s(\boldsymbol{x}, t)\frac{\delta F}{\delta M_\beta(\boldsymbol{x}, t)}\right\rangle\right)\mathrm{d}^3x \tag{2.46}$$

$$\{F, G\}_{\mathrm{E}}^{M}=-\int_{\Omega'}\left(\frac{\delta F}{\delta M_\gamma(\boldsymbol{x}, t)}\frac{\partial}{\partial x_\beta}\left\langle M_\gamma(\boldsymbol{x}, t)\frac{\delta G}{\delta M_\beta(\boldsymbol{x}, t)}\right\rangle - \frac{\delta G}{\delta M_\gamma(\boldsymbol{x}, t)}\frac{\partial}{\partial x_\beta}\left\langle M_\gamma(\boldsymbol{x}, t)\frac{\delta F}{\delta M_\beta(\boldsymbol{x}, t)}\right\rangle\right)\mathrm{d}^3x \tag{2.47}$$

and

$$\begin{aligned}\{F, G\}_{\mathrm{E}}^{C}=&-\int_{\Omega'}\left(\frac{\delta F}{\delta C_{\alpha\beta}}\frac{\partial}{\partial x_\gamma}\left\langle C_{\alpha\beta}\frac{\delta G}{\delta M_\gamma}\right\rangle-\frac{\delta G}{\delta C_{\alpha\beta}}\frac{\partial}{\partial x_\gamma}\left\langle C_{\alpha\beta}\frac{\delta F}{\delta M_\gamma}\right\rangle\right)\mathrm{d}^3x\\&-\int_{\Omega'}C_{\gamma\alpha}\left(\frac{\delta G}{\delta C_{\alpha\beta}}\frac{\partial}{\partial x_\gamma}\left\langle\frac{\delta F}{\delta M_\beta}\right\rangle-\frac{\delta F}{\delta C_{\alpha\beta}}\frac{\partial}{\partial x_\gamma}\left\langle\frac{\delta G}{\delta M_\beta}\right\rangle\right)\mathrm{d}^3x\\&-\int_{\Omega'}C_{\gamma\beta}\left(\frac{\delta G}{\delta C_{\alpha\beta}}\frac{\partial}{\partial x_\gamma}\left\langle\frac{\delta F}{\delta M_\alpha}\right\rangle-\frac{\delta F}{\delta C_{\alpha\beta}}\frac{\partial}{\partial x_\gamma}\left\langle\frac{\delta C}{\delta M_\alpha}\right\rangle\right)\mathrm{d}^3x.\end{aligned} \tag{2.48}$$

By summing the four sub-brackets via equation (2.42), we thus arrive at the spatial counterpart of the Poisson bracket in the material description:

$$\begin{aligned}\{F, G\}_{\mathrm{E}}=&-\int_{\Omega'}\left(\frac{\delta F}{\delta\rho}\nabla_\beta\left(\frac{\delta G}{\delta M_\beta}\rho\right)-\frac{\delta G}{\delta\rho}\nabla_\beta\left(\frac{\delta F}{\delta M_\beta}\rho\right)\right)\mathrm{d}^3x\\&-\int_{\Omega'}\left(\frac{\delta F}{\delta M_\gamma}\nabla_\beta\left(\frac{\delta G}{\delta M_\beta}M_\gamma\right)-\frac{\delta G}{\delta M_\gamma}\nabla_\beta\left(\frac{\delta F}{\delta M_\beta}M_\gamma\right)\right)\mathrm{d}^3x\\&-\int_{\Omega'}\left(\frac{\delta F}{\delta s}\nabla_\beta\left(\frac{\delta G}{\delta M_\beta}s\right)-\frac{\delta G}{\delta s}\nabla_\beta\left(\frac{\delta F}{\delta M_\beta}s\right)\right)\mathrm{d}^3x\\&-\int_{\Omega'}\left(\frac{\delta F}{\delta C_{\alpha\beta}}\nabla_\gamma\left(C_{\alpha\beta}\frac{\delta G}{\delta M_\gamma}\right)-\frac{\delta G}{\delta C_{\alpha\beta}}\nabla_\gamma\left(C_{\alpha\beta}\frac{\delta F}{\delta M_\gamma}\right)\right)\mathrm{d}^3x\\&-\int_{\Omega'}C_{\gamma\alpha}\left(\frac{\delta G}{\delta C_{\alpha\beta}}\nabla_\gamma\left(\frac{\delta F}{\delta M_\beta}\right)-\frac{\delta F}{\delta C_{\alpha\beta}}\nabla_\gamma\left(\frac{\delta G}{\delta M_\beta}\right)\right)\mathrm{d}^3x\\&-\int_{\Omega'}C_{\gamma\beta}\left(\frac{\delta G}{\delta C_{\alpha\beta}}\nabla_\gamma\left(\frac{\delta F}{\delta M_\alpha}\right)-\frac{\delta F}{\delta C_{\alpha\beta}}\nabla_\gamma\left(\frac{\delta G}{\delta M_\alpha}\right)\right)\mathrm{d}^3x\end{aligned} \tag{2.49}$$

where we have replaced $\partial/\partial x_\beta$ with ∇_β. This bracket is bilinear, antisymmetric and satisfies the Jacobi identity. When integrated by parts, the first three integrals take the form that was originally postulated for ideal fluid flow by Morrison and Greene (1980, p 793); however, these three integrals have appeared in a number of places (e.g. Holm and Kupershmidt 1983). The last three integrals represent the elastic contribution to the Poisson bracket, and appear in general form in Marsden *et al* (1984, p 171). The incompressible version of this bracket was postulated by Grmela (1988, p 83), but is derived here for the first time.

Now that we have the proper form of the Poisson bracket in terms of spatial variables, we only need to specify the spatial counterpart of the Hamiltonian, equation (2.15). If we consider the arbitrary spatial observation point, x, with volume element d^3x, which corresponds to the material coordinate Y, with volume element d^3Y at time t, then we can transfer the Hamiltonian of equation (2.15) directly over to the spatial coordinate at this instant through a direct change of variables:

$$H[\rho, \boldsymbol{M}, s, \mathbf{C}] = \int_{\Omega'} \left(\frac{|M(x,t)|^2}{2\rho(x,t)} + \rho(x,t)e_p^v(x,t) + \rho(x,t)\hat{U}[\rho, s, \mathbf{C}] \right) \mathrm{d}^3x. \tag{2.50}$$

In order to derive the Eulerian form of the equations of motion, we only need to consider an arbitrary functional, F, of the dynamical variables ρ, $\boldsymbol{M}$, s, and $\mathbf{C}$:

$$F[\rho, \boldsymbol{M}, s, \mathbf{C}] = \int_{\Omega'} f(\rho, \boldsymbol{M}, s, \mathbf{C})\, \mathrm{d}^3x. \tag{2.51}$$

Hence we can write the dynamical equation for this functional as

$$\frac{\mathrm{d}F}{\mathrm{d}t} = \int_{\Omega'} \left(\frac{\delta F}{\delta \rho}\frac{\partial \rho}{\partial t} + \frac{\delta F}{\delta M_\alpha}\frac{\partial M_\alpha}{\partial t} + \frac{\delta F}{\delta s}\frac{\partial s}{\partial t} + \frac{\delta F}{\delta C_{\alpha\beta}}\frac{\partial C_{\alpha\beta}}{\partial t} \right) \mathrm{d}^3x. \tag{2.52}$$

Analogously to equation (2.17), we can also write the dynamical equation for F in terms of the bracket (2.49) as

$$\frac{\mathrm{d}F}{\mathrm{d}t} = \{F, H\}_{\mathrm{E}}. \tag{2.53}$$

The two versions of the dynamical equation for F, equations (2.52) and (2.53), can now be compared directly to obtain the equations of motion for the spatial variables:

$$\frac{\partial \rho}{\partial t} = -\nabla_\beta(v_\beta \rho) \tag{2.54}$$

$$\frac{\partial s}{\partial t} = -\nabla_\beta(v_\beta s) \tag{2.55}$$

$$\rho\frac{\partial v_\alpha}{\partial t} = -\rho v_\beta \nabla_\beta v_\alpha - \rho\nabla_\alpha e_p^v - \nabla_\alpha p + 2\nabla_\gamma\left(C_{\gamma\beta}\frac{\partial u}{\partial C_{\alpha\beta}} \right) \tag{2.56}$$

and

$$\frac{\partial C_{\alpha\beta}}{\partial t} = -\nabla_\gamma(v_\gamma C_{\alpha\beta}) + C_{\gamma\alpha}\nabla_\gamma v_\beta + C_{\gamma\beta}\nabla_\gamma v_\alpha \tag{2.57}$$

where p is defined in terms of the internal energy density as

$$p \equiv \rho\frac{\partial u}{\partial \rho} + s\frac{\partial u}{\partial s} + C_{\alpha\beta}\frac{\partial u}{\partial C_{\alpha\beta}} - u. \tag{2.58}$$

(It is easy to show the relation between equations (2.20) and (2.58).) Grmela (1989, p 4386) incorporates a dependence upon the velocity into his equivalent pressure definition for complex fluids, but this appears to us to be erroneous. It does not seem plausible that the velocity should appear in a local equilibrium relation, such as this one. The elasticity of the medium defines an extra stress in the momentum equation, expressed by the last term in equation (2.56). This is the usual extra stress from continuum mechanics. The structure evolution equation (2.57) is a materially objective time derivative—Oldroyd's B derivative (1950, p 538)—and is commonly called the upper-convected derivative. This derivative plays the major role in modern rheological theories. Since the instant t is completely arbitrary, equations (2.54)–(2.57) represent *the equations of motion in the spatial description at all times.*

We can also obtain the energy equations for the compressible elastic medium quite easily. Since we know that

$$\frac{\partial u(\rho, s, \mathbf{C})}{\partial t}=\frac{\partial u}{\partial \rho}\frac{\partial \rho}{\partial t}+\frac{\partial u}{\partial s}\frac{\partial s}{\partial t}+\frac{\partial u}{\partial C_{\alpha\beta}}\frac{\partial C_{\alpha\beta}}{\partial t} \tag{2.59}$$

we can substitute into this expression equation (2.54), (2.55) and (2.57) to obtain, after some rearranging,

$$\frac{\partial u}{\partial t}=-\nabla_\alpha(v_\alpha u)-p\nabla_\alpha v_\alpha+2C_{\gamma\alpha}\frac{\partial u}{\partial C_{\alpha\beta}}\nabla_\gamma v_\beta \tag{2.60}$$

which is the equation for the internal energy. Likewise, we find the kinetic energy equation by dotting $\boldsymbol{v}$ with equation (2.56):

$$\rho\frac{\partial(\frac{1}{2}v^2)}{\partial t}=-\rho v_\alpha\nabla_\alpha(\tfrac{1}{2}v^2)+v_\alpha\nabla_\alpha p-\rho v_\alpha\nabla_\alpha e_{\rm p}^{v}+2v_\alpha\nabla_\gamma\left(C_{\gamma\beta}\frac{\partial u}{\partial C_{\alpha\beta}}\right) \tag{2.61}$$

and, finally, the energy equation may be obtained by the addition of equations (2.60) and (2.61):

$$\frac{\partial e}{\partial t}=-\nabla_\alpha(v_\alpha e)-\nabla_\alpha(pv_\alpha)-\rho v_\alpha\nabla_\alpha e_{\rm p}^{v}+2\nabla_\gamma\left(v_\alpha C_{\gamma\beta}\frac{\partial u}{\partial C_{\alpha\beta}}\right) \tag{2.62}$$

where $e=e_{\rm k}+u$.

Grmela (1989, p 4379) used a projection mapping to obtain the bracket representation of the energy equations. Although this procedure nicely illustrates the consistency of the bracket scheme, the added complexity detracts from the beauty of the theory. The entire system of dynamical equations can be unequivocally described through the bracket expression of equation (2.49).

3. The hydrodynamics of dissipative media

As alluded to in the introduction, the Poisson bracket can only describe conservative effects; however, most systems contain dissipative mechanisms due to irreversible processes, in particular, the case at hand of viscoelastic media. These dissipative phenomena arise due to the interactions of many degrees of freedom. Indeed, in the simplest possible case, that of a system of coupled, harmonic oscillators, it has been shown that (focusing on one individual oscillator interacting with the rest) it is possible to obtain a dynamic behaviour corresponding to a dissipative Brownian particle (Ford *et al* 1965) or to a damped harmonic oscillator (Yurke 1984, Harris 1990 and references therein). However, the requirement for dissipative behaviour to be observed (in the sense that the oscillator 'forgets' its initial conditions) is that a continuous frequency spectrum of modes exist in the system, implying an infinite number of degrees of freedom (Dinariev and Mosolov 1988). We have opted instead to incorporate dissipative properties directly into the generalized bracket by introducing the dissipation bracket.

The properties of a dissipation bracket, denoted $[F, G]$ for two arbitrary functionals, are only recently becoming clear. A form discussed by Kaufman (1984) and Grmela (1985) suggests that it should be bilinear, symmetric, act as a derivative and be a non-negative quantity. Beris and Edwards (1990a) proposed a similar form for *incompressible* viscoelastic fluids; however, in Edwards and Beris (1990) a new phenomenological dissipation bracket was proposed which expressed the equations of motion in Hamiltonian form for both single-component and multicomponent fluid systems. Also, Grmela (1989) proposed a nonlinear dissipative form.

The form of the dissipation bracket clearly depends upon the Hamiltonian (energy) assumed for the system as well as the dynamical variables of the problem formulation. If the Hamiltonian represents only the conservative part of the system energy (i.e. the kinetic and potential energies), as was the case in Grmela (1988) and Beris and Edwards (1990a, b) for incompressible systems, then the dissipation must clearly be a non-positive form, and also symmetric. If we define a generalized bracket according to the dynamical relation

$$\frac{\mathrm{d}F}{\mathrm{d}t}=\{(F, H)\}\equiv\{F, H\}+[F, H] \tag{3.1}$$

in the spatial description (dropping the subscript E on the Poisson bracket), then $[e_k+e_p, H]\leqslant 0$, since $\mathrm{d}H/\mathrm{d}t\leqslant 0$.

In the present work, the Hamiltonian represents the total energy of the system and, as such, it has to remain constant in time according to the first law of thermodynamics. Consequently, $\mathrm{d}H/\mathrm{d}t=0$, and therefore $[H, H]=0$. This requirement on the dissipation bracket guarantees that the energy will be conserved in the solution of the resulting equations. Of course, any associated entropy production must be positive, which requires that $\mathrm{d}S/\mathrm{d}t=[S, H]\geqslant 0$, where $S=\int s\,\mathrm{d}V$ is the entropy functional. For this system, we also require that the total mass is conserved: $\mathrm{d}W/\mathrm{d}t=\{W, H\}+[W, H]=0$, where $W=\int\rho\,\mathrm{d}V$. It is also obvious from section 2 that the dissipation bracket must be linear in F, but it may in general be nonlinear in H (or G).

Using the above properties, we may write the most general possible expression for the dissipation bracket as

$$[F, H]=-\Xi\left(L\left[\frac{\delta F}{\delta\boldsymbol{\omega}}, \nabla\frac{\delta F}{\delta \boldsymbol{w}}\right]; \frac{\delta H}{\delta \boldsymbol{w}}, \nabla\frac{\delta H}{\delta \boldsymbol{w}}\right)+\frac{\delta F}{\delta s}\Psi\left(\frac{\delta H}{\delta \boldsymbol{w}}, \nabla\frac{\delta H}{\delta \boldsymbol{w}}\right) \tag{3.2}$$

where the $L[\cdot]$ denotes that Ξ is linear with respect to $\cdot$, $\boldsymbol{w}=(a, b, c, \ldots, \boldsymbol{M}, s)$ and $\boldsymbol{\omega}=(a, b, c, \ldots, \boldsymbol{M})$; i.e. $\boldsymbol{\omega}=\boldsymbol{w}-s$, s being the entropy density. Thus equation (3.2) is the most general expression where we have pulled out the linear dependence on $\delta F/\delta s$ and written it separately. (Note that the minus sign in front of Ξ appears merely for later convenience.) The quantities $\delta H/\delta \boldsymbol{w}$ and $\nabla(\delta H/\delta \boldsymbol{w})$ represent the system affinities or thermodynamic forces; the former being associated with relaxational phenomena and the latter with the fluxes of the primary variables, e.g. $\nabla\boldsymbol{u}$, the velocity gradient tensor field.

Now if we use the properties described above, we can learn a little more about this expression. For instance, using the property that $[S, H] \geq 0$ (which indicates why we broke the expression up the way we did), we immediately see that Ψ must be a convex, non-negative function. Furthermore, via the conservation of energy, $[H, H] = 0$, we see that

$$\Psi = \frac{1}{\delta H/\delta s} \Xi\left(L\left[\frac{\delta H}{\delta \boldsymbol{\omega}}, \nabla \frac{\delta H}{\delta \boldsymbol{w}}\right]; \frac{\delta H}{\delta \boldsymbol{w}}, \nabla \frac{\delta H}{\delta \boldsymbol{w}} \right). \tag{3.3}$$

Ψ is a convex function, with an absolute minimum equal to zero at equilibrium where $\delta H/\delta \boldsymbol{\omega} = 0$ and $\nabla(\delta H/\delta \boldsymbol{w}) = \mathbf{0}$, i.e.

$$\Psi\left(\frac{\delta H}{\delta \boldsymbol{\omega}} = \mathbf{0}, \nabla \frac{\delta H}{\delta \boldsymbol{w}} = \mathbf{0} \right) = 0 \tag{3.4}$$

with

$$\left. \frac{\partial \Psi}{\partial(\delta H/\delta \boldsymbol{\omega})} \right|_{(0,0)} = \mathbf{0} \qquad \left. \frac{\partial \Psi}{\partial(\nabla[\delta H/\delta \boldsymbol{w}])} \right|_{(0,0)} = \mathbf{0} \tag{3.5}$$

and non-negative definite second derivatives. Using the thermodynamic definition of the absolute temperature, $T \equiv \delta H/\delta s$, as well as equation (3.3), we can rewrite the dissipation bracket of equation (3.2) in terms of arbitrary G as well, as

$$[F, G] = -\Xi\left(L\left[\frac{\delta F}{\delta \boldsymbol{\omega}}, \nabla \frac{\delta F}{\delta \boldsymbol{w}}\right]; \frac{\delta G}{\delta \boldsymbol{w}}, \nabla \frac{\delta G}{\delta \boldsymbol{w}} \right) + \frac{1}{T} \frac{\delta F}{\delta s} \Xi\left(L\left[\frac{\delta G}{\delta \boldsymbol{\omega}}, \nabla \frac{\delta G}{\delta \boldsymbol{w}}\right]; \frac{\delta G}{\delta \boldsymbol{w}}, \nabla \frac{\delta G}{\delta \boldsymbol{w}} \right). \tag{3.6}$$

In this article, we restrict ourselves to systems which are close to equilibrium, i.e. when Ξ is linear with respect to G (or H) as well as F. Hence we can express Ξ in terms of the functional derivatives as

$$\Xi = \sum_{\substack{i,j\\k,l}} \left(A_{ij} \frac{\delta F}{\delta \omega_i} \frac{\delta G}{\delta w_i} + B_{ijk} \frac{\delta F}{\delta \omega_i} \nabla_k \frac{\delta G}{\delta w_j} + C_{ijk} \nabla_k \frac{\delta F}{\delta w_i} \frac{\delta G}{\delta w_j} + D_{ijkl} \nabla_k \frac{\delta F}{\delta w_i} \nabla_l \frac{\delta G}{\delta w_j} \right) \tag{3.7}$$

where $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$ are phenomenological coefficient matrices which, in general, may depend on $\delta G/\delta s$ and the primary variables of the system. One can determine constraints upon the phenomenological coefficients by considering the conservation of mass ($\mathrm{d}W/\mathrm{d}t = 0$) and material objectivity, as well as the microscopic time reversibility of the medium. In case of the latter, due to the symmetry of the bracket, one can arrive at the Onsager–Casimir reciprocal relations which equate the phenomenological coefficients of opposing fluxes, i.e. $A_{ij} = A_{ji}$. More specifically, due to the Onsager–Casimir arguments under the microscopic time reversal, it is assumed that (Woods 1975, p 159)

$$\boldsymbol{\Phi}^{ij}(\tau) = \eta_i \eta_j \boldsymbol{\Phi}^{ji}(-\tau) \tag{3.8}$$

where η_a and η_b are the parities of each affinity $\nabla(\delta H/\delta a)$ and $\nabla(\delta H/\delta b)$ under the time reversal, indicated by τ. This also leads to the requirement that $\phi^{ij}_{\alpha\beta} = \phi^{ji}_{\beta\alpha}$. Furthermore, the types of coupling are restricted by the Curie principle, i.e. that only affinities of the same tensorial character can couple with each other.

In the following subsections, we illustrate the use of this dissipation bracket (see equation (3.2)) in constructing thermodynamically consistent equations of motion for viscoelastic fluids. Although the conservative dynamical equations are quite standard, the dissipative fluid equations represent a new form for the Eulerian viscoelastic fluid equations. Throughout we assume that the Curie principle is valid. For the special cases of the single-component and multicomponent simple fluids, the reader is referred to Edwards and Beris (1990).

As described in section 2, the dynamical variables in elastic fluid dynamics are ρ, s, $\boldsymbol{M}$, and $\mathbf{C}$ (with corresponding dissipative affinities $\nabla[\delta H/\delta\rho]$, $\nabla[\delta H/\delta s]$, $\nabla[\delta H/\delta \boldsymbol{M}]$, $\delta H/\delta\mathbf{C}$, and $\nabla[\delta H/\delta\mathbf{C}]$). Thus the dissipation bracket for this case, involving all possible couplings not forbidden by the Curie principle, is

$$
\begin{aligned}
[F,G] = &-\int_{\Omega'} Q_{\alpha\beta\gamma\varepsilon}\left(\nabla_\alpha \frac{\delta F}{\delta M_\beta}\nabla_\gamma\frac{\delta G}{\delta M_\varepsilon}\right)\mathrm{d}^3x - \int_{\Omega'} A_{\alpha\beta}\left(\nabla_\alpha\frac{\delta F}{\delta\rho}\nabla_\beta\frac{\delta G}{\delta\rho}\right)\mathrm{d}^3x \\
&-\int_{\Omega'} B_{\alpha\beta}\left(\nabla_\alpha\frac{\delta F}{\delta s}\nabla_\beta\frac{\delta G}{\delta s}\right)\mathrm{d}^3x \\
&-\int_{\Omega'} D_{\alpha\beta}\left(\nabla_\alpha\frac{\delta F}{\delta\rho}\nabla_\beta\frac{\delta G}{\delta s}+\nabla_\alpha\frac{\delta F}{\delta s}\nabla_\beta\frac{\delta G}{\delta\rho}\right)\mathrm{d}^3x \\
&-\int_{\Omega'} R_{\alpha\beta\gamma\varepsilon\eta\upsilon}\left(\nabla_\alpha\frac{\delta F}{\delta C_{\beta\gamma}}\nabla_\varepsilon\frac{\delta G}{\delta C_{\eta\upsilon}}\right)\mathrm{d}^3x - \int_{\Omega'} \Lambda_{\alpha\beta\gamma\varepsilon}\left(\frac{\delta F}{\delta C_{\alpha\beta}}\frac{\delta G}{\delta C_{\gamma\varepsilon}}\right)\mathrm{d}^3x \\
&-\int_{\Omega'} P_{\alpha\beta\gamma\varepsilon}\left(\nabla_\alpha\frac{\delta F}{\delta M_\beta}\frac{\delta G}{\delta C_{\gamma\varepsilon}} - \nabla\left(\frac{\delta G}{\delta M_\beta}\frac{\delta F}{\delta C_{\gamma\varepsilon}}\right)\right)\mathrm{d}^3x \\
&+\int_{\Omega'} Q_{\alpha\beta\gamma\varepsilon}\frac{1}{T}\frac{\delta F}{\delta s}\left(\nabla_\alpha\frac{\delta G}{\delta M_\beta}\nabla_\gamma\frac{\delta G}{\delta M_\varepsilon}\right)\mathrm{d}^3x \\
&+\int_{\Omega'} A_{\alpha\beta}\frac{1}{T}\frac{\delta F}{\delta s}\left(\nabla_\alpha\frac{\delta G}{\delta\rho}\nabla_\beta\frac{\delta G}{\delta\rho}\right)\mathrm{d}^3x \\
&+\int_{\Omega'} B_{\alpha\beta}\frac{1}{T}\frac{\delta F}{\delta s}\left(\nabla_\alpha\frac{\delta G}{\delta s}\nabla_\beta\frac{\delta G}{\delta s}\right)\mathrm{d}^3x \\
&+2\int_{\Omega'} D_{\alpha\beta}\frac{1}{T}\frac{\delta F}{\delta s}\left(\nabla_\alpha\frac{\delta G}{\delta\rho}\nabla_\beta\frac{\delta G}{\delta s}\right)\mathrm{d}^3x \\
&+\int_{\Omega'} R_{\alpha\beta\gamma\varepsilon\eta\upsilon}\frac{1}{T}\frac{\delta F}{\delta s}\left(\nabla_\alpha\frac{\delta G}{\delta C_{\beta\gamma}}\nabla_\varepsilon\frac{\delta G}{\delta C_{\eta\upsilon}}\right)\mathrm{d}^3x \\
&+\int_{\Omega'} \Lambda_{\alpha\beta\gamma\varepsilon}\frac{1}{T}\frac{\delta F}{\delta s}\left(\frac{\delta G}{\delta C_{\alpha\beta}}\frac{\delta G}{\delta C_{\gamma\varepsilon}}\right)\mathrm{d}^3x
\end{aligned}
\tag{3.9}
$$

where we have used the reciprocal relations (3.8) and $\mathbf{Q}$, $\mathbf{P}$ and $\mathbf{R}$ satisfy additional objectivity relationships of the form $Q_{\alpha\beta\gamma\varepsilon} = Q_{\gamma\varepsilon\alpha\beta} = Q_{\beta\alpha\gamma\varepsilon} = Q^{\alpha\beta\varepsilon\gamma}$. In order for mass to be preserved, it is necessary that $\mathbf{A} = \mathbf{D} = \mathbf{0}$; however, this is not the case for the multicomponent simple fluid. (See Edwards and Beris (1990) for details.) The first integral represents the effects of viscosity upon the equations of motion, and the third represents the effects of heat conduction. Assuming that the fluid is isotropic, one can set $B_{\alpha\beta} = k\delta_{\alpha\beta}$ and $Q_{\alpha\beta\gamma\varepsilon} = \mu\delta_{\alpha\gamma}\delta_{\beta\varepsilon} + \mu\delta_{\alpha\varepsilon}\delta_{\beta\gamma} + \kappa'\delta_{\alpha\beta}\delta_{\gamma\varepsilon}$, where k is a thermal conductivity, μ is a shear viscosity and κ' is a function of the bulk viscosity, κ, and the shear viscosity: $\kappa' = \kappa - 2\mu/3$.

In the above expression, the fifth integral represents the effects of the translational diffusivity of the polymer molecules, i.e. the spatial variations due to an inhomogeneous concentration. An expression for the phenomenological tensor **R** is given by Edwards *et al* (1990) for the special case of rigid rods. The sixth integral represents the effects of relaxational phenomena, where **Λ** can be interpreted as an inverse relaxation time. (See Beris and Edwards (1990a, pp 67, 68) for details.) The seventh integral term represents the effects of non-affine motion in the system. The minus sign in the integrand of this term arises according to equation (3.8) since the affinities involved have opposite parities. Note that this term does not contribute to the entropy production.

Substitution of these expressions into the dissipation bracket (3.9) yields the following dissipative evolution in terms of the dynamical variables:

$$\frac{\partial \rho}{\partial t}=-\nabla_\beta(v_\beta \rho) \tag{3.10a}$$

$$\frac{\partial s}{\partial t}=-\nabla_\gamma(v_\gamma s)+\frac{1}{T}\nabla_\gamma(kT\nabla_\gamma T)+\frac{1}{T}[\mu(\nabla_\gamma v_\alpha+\nabla_\alpha v_\gamma)+\kappa'\delta_{\alpha\gamma}\nabla_\varepsilon v_\varepsilon]\nabla_\alpha v_\gamma$$
$$+\frac{1}{T}R_{\alpha\beta\gamma\varepsilon\eta\upsilon}\nabla_\alpha\frac{\partial u}{\partial C_{\beta\gamma}}\nabla_\varepsilon\frac{\partial u}{\partial C_{\eta\upsilon}}+\frac{1}{T}\Lambda_{\alpha\beta\gamma\varepsilon}\frac{\partial u}{\partial C_{\alpha\beta}}\frac{\partial u}{\partial C_{\gamma\varepsilon}} \tag{3.10b}$$

$$\rho\frac{\partial v_\alpha}{\partial t}=-\rho v_\beta\nabla_\beta v_\alpha-\rho\nabla_\alpha e_{\mathrm{p}}^{v}-\nabla_\alpha p+\nabla_\gamma[\mu(\nabla_\gamma v_\alpha+\nabla_\alpha v_\gamma)+\kappa'\delta_{\alpha\gamma}\nabla_\varepsilon v_\varepsilon]$$
$$+\nabla_\beta\left(P_{\alpha\beta\gamma\varepsilon}\frac{\partial u}{\partial C_{\gamma\varepsilon}}\right)+2\nabla_\gamma\left(C_{\gamma\beta}\frac{\partial u}{\partial C_{\beta\alpha}}\right) \tag{3.10c}$$

and

$$\frac{\partial C_{\alpha\beta}}{\partial t}=-\nabla_\gamma(v_\gamma C_{\alpha\beta})+C_{\gamma\alpha}\nabla_\gamma v_\beta+C_{\gamma\beta}\nabla_\gamma v_\alpha+P_{\alpha\beta\gamma\varepsilon}\nabla_\gamma v_\varepsilon$$
$$-\Lambda_{\alpha\beta\gamma\varepsilon}\frac{\partial u}{\partial C_{\gamma\varepsilon}}+\nabla_\gamma\left[R_{\alpha\beta\gamma\varepsilon\eta\upsilon}\nabla_\upsilon\left(\frac{\partial u}{\partial C_{\varepsilon\eta}}\right)\right]. \tag{3.10d}$$

Of course, the no-penetration boundary condition must be replaced with the no-slip condition. As in section 2, the energy equation can then be obtained as

$$\frac{\partial e}{\partial t}=-\nabla_\alpha(v_\alpha e)-\nabla_\alpha(pv_\alpha)-\rho v_\alpha\nabla_\alpha e_{\mathrm{p}}^{v}+2\nabla_\gamma\left(v_\alpha C_{\gamma\beta}\frac{\partial u}{\partial C_{\alpha\beta}}\right)+\nabla_\alpha(kT\nabla_\alpha T)$$
$$+2\nabla_\gamma\left(v_\alpha C_{\gamma\beta}\frac{\partial u}{\partial C_{\alpha\beta}}\right)+\nabla_\gamma\{[\mu(\nabla_\gamma v_\alpha+\nabla_\alpha v_\gamma)+\kappa'\delta_{\alpha\gamma}\nabla_\varepsilon v_\varepsilon]v_\alpha\}$$
$$+\nabla_\beta\left(v_\alpha P_{\alpha\beta\gamma\varepsilon}\frac{\partial u}{\partial C_{\gamma\varepsilon}}\right)+\nabla_\gamma\left[\frac{\partial u}{\partial C_{\alpha\beta}}R_{\alpha\beta\gamma\varepsilon\eta\upsilon}\nabla_\upsilon\left(\frac{\partial u}{\partial C_{\varepsilon\eta}}\right)\right]. \tag{3.11}$$

This completes the development of the dissipative hydrodynamic equations for viscoelastic media.

Yet, we may obtain one more piece of information from the dissipation bracket, i.e. we may use it to obtain the entropy inequality for the system. Since we know that $\mathrm{d}S/\mathrm{d}t\geqslant 0$, via the bracket of equation (3.9), we arrive at the following inequality which must be satisfied for the viscoelastic system:

$$Q_{\alpha\beta\gamma\varepsilon}\nabla_\alpha b_\beta\nabla_\gamma v_\varepsilon+B_{\alpha\beta}\nabla_\alpha T\nabla_\beta T$$
$$+R_{\alpha\beta\gamma\varepsilon\eta\upsilon}\nabla_\alpha\frac{\delta H}{\delta C_{\eta\gamma}}\nabla_\varepsilon\frac{\delta H}{\delta C_{\eta\upsilon}}+\Lambda_{\alpha\beta\gamma\varepsilon}\frac{\delta H}{\delta C_{\alpha\beta}}\frac{\delta H}{\delta C_{\gamma\varepsilon}}\geqslant 0. \tag{3.12}$$

Many well known viscoelastic models can now be expressed in the Hamiltonian formalism, and thus extended consistently to compressible situations. (For incompressible fluids, this was recently attempted by Beris and Edwards (1990b).) The phenomenological tensors in the above expressions have been deliberately left unspecified to make the transport equations as general as possible. It is in these parameters that the various viscoelastic fluid models must differ from each other, since every model must have the above underlying structure in order to be thermodynamically consistent. Aside from the symmetry constraints upon these phenomenological tensors as indicated above, one is also limited by the mathematical constraints imposed upon the second-order tensor **C**, for instance, the Cayley–Hamilton theorem.

In particular, let us discuss briefly two of the simplest viscoelastic models, which concern a spring and a dashpot: the spring and dashpot in parallel (Voigt model) and the spring and dashpot in series (Maxwell model). (A combination of these two models gives the Oldroyd-B fluid.) Although originally derived for an incompressible fluid, these two models may easily be extended to compressible situations via the above development. (For the incompressible Hamiltonian formulation of these models, see Beris and Edwards (1990a, section IV).) Here, we let **C** be a general, structural-density tensor with units of mass/length. The Voigt model may then be obtained in the above development by setting $\mathbf{B}=\mathbf{P}=\mathbf{\Lambda}=\mathbf{R}=\mathbf{0}$. with the Hamiltonian provided by equation (2.50) with $u=a+Ts$, where the Helmholtz free energy density is defined as

$$a=(\nu k_B T/2)\,\mathrm{tr}\,\mathbf{C}'-(\nu\rho k_B T/2)\ln[\det(\mathbf{C}'/\rho)]+\tilde{a}(\rho,s) \tag{3.13}$$

where $\mathbf{C}'=(K/k_B T)\mathbf{C}$ is a dimensionless structural parameter replacing **C** in the bracket expressions (2.49) and (3.4), K is the Hookean spring constant, ν is the number of springs per unit mass of material and k_B is Boltzmann's constant. The first term on the right-hand side of equation (3.13) describes the elastic extension of the constituent spring, the second term describes the Boltzmann entropy, and the last term describes the dependence of the free energy upon the mass density and entropy density. The Maxwell model can similarly be obtained with the same Hamiltonian but with dissipative parameters $\mathbf{Q}=\mathbf{B}=\mathbf{P}=\mathbf{R}=\mathbf{0}$ and

$$\Lambda_{\alpha\beta\gamma\varepsilon}=(1/2\nu K\lambda)(C'_{\alpha\gamma}\delta_{\beta\varepsilon}+C'_{\beta\varepsilon}\delta_{\alpha\gamma}+C'_{\beta\gamma}\delta_{\alpha\varepsilon}+C'_{\alpha\varepsilon}\delta_{\beta\gamma}) \tag{3.14}$$

where λ is the system relaxation time. Other more complicated models can be expressed in Hamiltonian form in a similar manner, following Beris and Edwards (1990b).

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